Math 33A:

<u>Today</u>: Eigenvalues, Eigenvectors

<u>OH</u>: Thurs, 1-2 pm, Zoom

<u>HW 3</u>: Due tomorrow (7/24), 11:59 pm

Eigenvectors / Eigenvalues : A square matrix, $n \times n$

Idea: We want to see for which vectors a matrix A

"acts simply"

$\hookrightarrow$ $A\vec{v} = \lambda\vec{v}$, λ a scalar, $\vec{v} \neq \vec{0}$

Such a vector v is called an <u>eigenvector</u>

And the corresp. λ is an <u>eigenvalue</u>

Rmk: If $\lambda = 0$, looking at $Av = 0v = 0$

$\Rightarrow$ $\ker(A)$.

$$Av = \lambda v$$

$$\Updownarrow$$

$$Av - \lambda v = \vec{0}$$

$$\Updownarrow$$

$$Av - \lambda I v = \vec{0}$$

$$\Updownarrow$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

$$\vec{v} \in \ker(A - \lambda I)$$

If $A - \lambda I$ has a non-zero kernel, it must be non-invertible.

This happens if & only if

$$\underbrace{\det(A - \lambda I)}_{\text{polynomial in terms of } \lambda} = 0$$

Characteristic polynomial of A

We want to find all eigenvalues & eigenvectors of A:

$$A = \begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ -3 & 0 & -1 \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ -3 & 0 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7-\lambda & 0 & 3 \\ -3 & 2-\lambda & -3 \\ -3 & 0 & -1-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} 7-\lambda & 0 & 3 \\ -3 & 2-\lambda & -3 \\ -3 & 0 & -1-\lambda \end{pmatrix} \begin{matrix} 7-\lambda & 0 \\ -3 & 2-\lambda \\ -3 & 0 \end{matrix}$$

$$(7-\lambda)(2-\lambda)(-1-\lambda)+0+0$$

$$-(-3)(2-\lambda)(3)$$

$$(2-\lambda)\left[(7-\lambda)(-1-\lambda)+9\right]=(2-\lambda)\left(\lambda^2-7\lambda+\lambda-7+9\right)$$

$$=(2-\lambda)(\lambda^2\underbrace{-6\lambda+2}_{})=0$$

$$ax^2+bx+c=0$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\lambda=\frac{6\pm\sqrt{36-8}}{2}=3\pm\frac{\sqrt{28}}{2}$$

$$=3\pm\sqrt{7}$$

$$\Rightarrow\quad\lambda=2,\ 3+\sqrt{7},\ 3-\sqrt{7}$$

$$(2-\lambda)(\lambda^2-6\lambda+2) = -\lambda^3+6\lambda^2-2\lambda+2\lambda^2-12\lambda+4$$
$$= -\lambda^3+8\lambda^2-14\lambda+4 = (\lambda-2)(\lambda-(3+\sqrt{7}))(\lambda-(3-\sqrt{7}))$$

Rational root thm: If a polynomial has a rational $\left(\frac{a}{b},\ a, b \text{ integers}\right)$ root, then b is a factor of the leading coefficient & a is a factor of the constant term. (factor - up to sign)

$(\lambda-2)^3(\lambda-1)^2$

2 has a.m. of 3

1 has a.m. of 2

Leading coefficient: (-1), ± 1

Constant term: 4, $\pm 1, \pm 2, \pm 4$

$$\frac{a}{b} = \frac{\pm 1, \pm 2, \pm 4}{\pm 1} = \pm 1, \pm 2, \pm 4$$

Once we see 2 is a root, we also know $(\lambda-2)$ is a factor of so we can use polynomial long division to factor it out

$$\lambda-2\,\overline{\big)\,-\lambda^3+8\lambda^2-14\lambda+4}$$

$$A = \begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ -3 & 0 & -1 \end{pmatrix}$$

$$\lambda = 2,\ 3+\sqrt{7},\ 3-\sqrt{7}$$

$$\ker(A - \lambda I)$$

<u>$\lambda = 2$</u>: $A - 2I$

$$\begin{pmatrix} 5 & 0 & 3 \\ -3 & 0 & -3 \\ -3 & 0 & -3 \end{pmatrix} \begin{array}{l} /5 \\ \\ -(\mathrm{II}) \end{array} \longrightarrow \begin{pmatrix} 1 & 0 & 3/5 \\ -3 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \\ +3(\mathrm{I}) \\ \frac{-15}{5} \end{array}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 3/5 \\ 0 & 0 & -6/5 \\ 0 & 0 & 0 \end{pmatrix}$$

x_2 free

$x_1 + 3/5\, x_3 = 0$

$x_1 + 0 = 0$

$x_1 = 0$

$-\frac{6}{5} x_3 = 0 \Rightarrow x_3 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad x_2 \text{ free} \Bigg\} \quad \ker(A - \lambda I) = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

eigenspace for $\lambda = 2$.

$$A \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2t \\ 0 \end{bmatrix} \quad \text{for any scalar } t$$

$\downarrow$

$\lambda = 2$ has a g.m. of 1

E_λ = the eigenspace for an eigenvalue λ

$\dim(E_\lambda) \rightarrow$ called the <u>geometric multiplicity</u> of λ

The # of times an eigenvalue λ shows up in the characteristic poly. is the <u>algebraic multiplicity</u> of λ

For an eigenvalue λ, $\boxed{\text{g.m. of } \lambda \leq \text{a.m. of } \lambda}$

$\underline{\lambda = 3 + \sqrt{7}}$:

$$\begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ -3 & 0 & -1 \end{pmatrix} \xrightarrow{A - \lambda I} \begin{pmatrix} 4-\sqrt{7} & 0 & 3 \\ -3 & -1-\sqrt{7} & -3 \\ -3 & 0 & -4-\sqrt{7} \end{pmatrix} /4-\sqrt{7} \sim$$

$$\begin{pmatrix} 1 & 0 & \boxed{3/4-\sqrt{7}} \\ -3 & -1-\sqrt{7} & -3 \\ -3 & 0 & -4-\sqrt{7} \end{pmatrix} \begin{matrix} \\ +3(I) \\ +3(I) \end{matrix}$$

$$= \frac{4+\sqrt{7}}{3}$$

$$\frac{3}{4-\sqrt{7}} \cdot \frac{4+\sqrt{7}}{4+\sqrt{7}} = \frac{12+3\sqrt{7}}{\underbrace{\cancel{16-7}}_{9}} = \frac{4+\sqrt{7}}{3}$$

$$\begin{pmatrix} 1 & 0 & \frac{4+\sqrt{7}}{3} \\ 0 & -1-\sqrt{7} & 1+\sqrt{7} \\ 0 & 0 & 0 \end{pmatrix} /-1-\sqrt{7}$$

$$\to \begin{pmatrix} 1 & 0 & \frac{4+\sqrt{7}}{3} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 = \frac{-4-\sqrt{7}}{3} x_3$

$x_2 = x_3$

x_3 free

g.m. of $3+\sqrt{7} = 1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{-4-\sqrt{7}}{3} \\ 1 \\ 1 \end{bmatrix}$$

$\underline{\lambda = 3 - \sqrt{7}}$:

$$\begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ -3 & 0 & -1 \end{pmatrix} \longrightarrow \begin{array}{c} \begin{pmatrix} 4+\sqrt{7} & 0 & 3 \\ -3 & -1+\sqrt{7} & -3 \\ -3 & 0 & -4+\sqrt{7} \end{pmatrix} \end{array} \begin{array}{l} /\, 4+\sqrt{7} \\ \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & 3/\cancel{4+\sqrt{7}} \quad \dfrac{4-\sqrt{7}}{3} \\ -3 & -1+\sqrt{7} & -3 \\ -3 & 0 & -4+\sqrt{7} \end{pmatrix} \begin{array}{l} \\ +3(I) \\ +3(I) \end{array} \longrightarrow \begin{pmatrix} 1 & 0 & \dfrac{4-\sqrt{7}}{3} \\ 0 & -1+\sqrt{7} & 1-\sqrt{7} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \frac{4-\sqrt{7}}{3} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} \frac{-4+\sqrt{7}}{3} \\ 1 \\ 1 \end{bmatrix}}$$